

# Actions of groups of diffeomorphisms on one-manifolds by $C^1$ diffeomorphisms

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ABSTRACT. Denote by  $\text{Diff}_c^r(M)_0$  the identity component of the group of the compactly supported  $C^r$  diffeomorphisms of a connected  $C^\infty$  manifold  $M$ . We show that if  $\dim(M) \geq 2$  and  $r \neq \dim(M) + 1$ , then any homomorphism from  $\text{Diff}_c^r(M)_0$  to  $\text{Diff}^1(\mathbb{R})$  or  $\text{Diff}^1(S^1)$  is trivial.

## 1. Introduction

É. Ghys [G] asked if the group of diffeomorphisms of a manifold admits a nontrivial action on a lower dimensional manifold. A break through towards this problem was obtained by K. Mann [M] for one dimensional target manifolds. Let  $M$  be a connected  $C^\infty$  manifold without boundary, compact or not. For  $r = 0, 1, 2, \dots, \infty$ , denote by  $\text{Diff}_c^r(M)_0$  the identity component of the group of the compactly supported  $C^r$  diffeomorphisms (homeomorphisms for  $r = 0$ ) of  $M$ .

THEOREM 1.1. (K. Mann) *Any homomorphism from  $\text{Diff}_c^r(M)_0$  to  $\text{Diff}^2(S^1)$  or to  $\text{Diff}^2(\mathbb{R})$  is trivial, provided  $\dim(M) \geq 2$  and  $r \neq \dim(M) + 1$ .*

For a simpler proof of this fact, see also [Ma2]. A natural question is whether it is possible to lower the differentiability of the target group. In fact for  $r = 0$ , E. Militon [Mi] obtained the final result.

THEOREM 1.2. (E. Militon) *Any homomorphism from  $\text{Diff}_c^0(M)_0$  to  $\text{Diff}^0(S^1)$  is trivial if  $\dim(M) \geq 2$ .*

Notice that  $\text{Diff}^0(\mathbb{R})$  can be considered to be a subgroup of  $\text{Diff}^0(S^1)$ . So we do not mention in the above theorem the case where the target group is  $\text{Diff}^0(\mathbb{R})$ .

Even for  $r \geq 1$ , we have:

CONJECTURE 1.3. Any homomorphism from  $\text{Diff}_c^r(M)_0$  to  $\text{Diff}^0(S^1)$  is trivial if  $\dim(M) \geq 2$ .

The purpose of this paper is to mark one step forward towards this conjecture.

THEOREM 1.4. *If  $\dim(M) \geq 2$  and  $r \neq \dim(M) + 1$ , any homomorphism from  $\text{Diff}_c^r(M)_0$  to  $\text{Diff}^1(S^1)$  or  $\text{Diff}^1(\mathbb{R})$  is trivial.*

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Frequent use of the simplicity of the group  $\text{Diff}_c^r(M)_0$  is made in the proof. The condition  $r \neq \dim(M) + 1$  is needed for it. As for Theorem 1.1, the proof is built upon a theorem of Kopell and Szekeres about  $C^2$  actions of abelian groups on a compact interval, while for Theorem 1.4, upon a theorem of Bonatti, Monteverde, Navas and Rivas about  $C^1$  actions of solvable Baumslag-Solitar groups on a compact interval.

By virtue of the fragmentation lemma, Theorem 1.4 reduces to:

**THEOREM 1.5.** *For  $n \geq 2$  and  $r \neq n + 1$ , any homomorphism from  $\text{Diff}_c^r(\mathbb{R}^n)_0$  to  $\text{Diff}^1(S^1)$  or  $\text{Diff}^1(\mathbb{R})$  is trivial.*

In Section 2, we show that the case of target group  $\text{Diff}^1(S^1)$  can be reduced to the case  $\text{Diff}^1(\mathbb{R})$ . In Sections 3 and 4, we establish fixed point results for certain subgroups of  $\text{Diff}_c^\infty(\mathbb{R}^n)_0$ . In Section 5, we prove Theorem 1.5 following an argument of E. Militon [Mi]. Finally we give some sporadic results for  $\text{Diff}^0(S^1)$  target in Section 6.

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## 2. Reduction to the case $\text{Diff}^1(\mathbb{R})$

In this section, we show that Theorem 1.5 for the target group  $\text{Diff}^1(S^1)$  is reduced to the case of  $\text{Diff}^1(\mathbb{R})$ .

**PROPOSITION 2.1.** *Let  $r \neq n + 1$  and  $n \geq 1$ . Assume that  $\Phi : \text{Diff}_c^r(\mathbb{R}^n)_0 \rightarrow \text{Diff}^0(S^1)$  is a nontrivial homomorphism. then the global fixed point set is nonempty:  $\text{Fix}(\Phi(\text{Diff}_c^r(\mathbb{R}^n)_0)) \neq \emptyset$ .*

This proposition enables us to conclude that the image of  $\Phi$  is contained in the group of the homeomorphisms of  $\mathbb{R}$ . In particular, Theorem 1.5 for the target group  $\text{Diff}^1(S^1)$  is reduced to the case of  $\text{Diff}^1(\mathbb{R})$ .

Denote  $\mathcal{G} = \text{Diff}_c^r(\mathbb{R}^n)_0$ . By the simplicity of the group  $\mathcal{G}$ , the homomorphism  $\Phi$  in the proposition is injective and its image is contained in  $\text{Diff}_+^0(S^1)$ , the group of the orientation preserving homeomorphisms.

Let  $B_0$  be the closed unit ball in  $\mathbb{R}^n$  centered at the origin. Define a family  $\mathcal{B}$  of the closed balls in  $\mathbb{R}^n$  by

$$\mathcal{B} = \{g(B_0) \mid g \in \mathcal{G}\}.$$

Also for  $B \in \mathcal{B}$ , let

$$\mathcal{G}(B) = \{g \in \mathcal{G} \mid \text{Supp}(g) \subset \text{Int}(B)\}.$$

To show Proposition 2.1, it is sufficient to show the following.

**PROPOSITION 2.2.** *For any  $B \in \mathcal{B}$ , the fixed point set  $\text{Fix}(\Phi(\mathcal{G}(B)))$  is nonempty.*

In fact, choose an increasing sequence of balls,  $\{B_k\}_{k \in \mathbb{N}} \subset \mathcal{B}$  such that  $\bigcup_k B_k = \mathbb{R}^n$ . Then we have  $\mathcal{G} = \bigcup_k \mathcal{G}(B_k)$  and  $\text{Fix}(\Phi(\mathcal{G})) = \bigcap_k \text{Fix}(\Phi(\mathcal{G}(B_k)))$ . Therefore by the compactness of  $S^1$ , Proposition 2.1 follows from Proposition 2.2.

Now for any  $B_1, B_2 \in \mathcal{B}$ , the groups  $\mathcal{G}(B_1)$  and  $\mathcal{G}(B_2)$  are conjugate in  $\mathcal{G}$ . Therefore their images  $\Phi(\mathcal{G}(B_1))$  and  $\Phi(\mathcal{G}(B_2))$  are conjugate in  $\text{Diff}_+^0(S^1)$ . They are simple. Moreover if  $B_1$  and  $B_2$  are disjoint, any element of  $\Phi(\mathcal{G}(B_1))$  commutes with any element of  $\Phi(\mathcal{G}(B_2))$ . Therefore Proposition 2.2 reduces to the following.

**PROPOSITION 2.3.** *Let  $G_1$  and  $G_2$  be simple nonabelian subgroups of  $\text{Diff}_+^0(S^1)$ . Assume that  $G_2$  is conjugate to  $G_1$  in  $\text{Diff}_+^0(S^1)$  and that any element of  $G_1$  commutes with any element of  $G_2$ . Then there is a global fixed point of  $G_1$ :  $\text{Fix}(G_1) \neq \emptyset$ .*

**PROOF.** Let  $X_2 \subset S^1$  be a minimal set of  $G_2$ . The set  $X_2$  is either a finite set, a Cantor set or the whole of  $S^1$ . If  $X_2$  is a singleton, then  $G_2$  admits a fixed point. Since  $G_1$  is conjugate to  $G_2$ , we have  $\text{Fix}(G_1) \neq \emptyset$ , as is required. So assume for contradiction that  $X_2$  is not a singleton.

First if  $X_2$  is a finite set which is not a singleton, we get a nontrivial homomorphism from  $G_2$  to a finite abelian group, contrary to the assumption of the simplicity. In the remaining case, it is well known, easy to show, that the minimal set is unique. That is,  $X_2$  is contained in any nonempty  $G_2$  invariant closed subset.

Let  $F_1$  be the subset of  $G_1$  formed by the elements  $g$  such that  $\text{Fix}(g) \neq \emptyset$ . Let us show that there is a nontrivial element in  $F_1$ . Assume the contrary. Then  $G_1$  acts freely on  $S^1$ . Consider the group  $\tilde{G}_1$  formed by any lift of any element of  $G_1$  to the universal covering space  $\mathbb{R} \rightarrow S^1$ . Now  $\tilde{G}_1$  acts freely on  $\mathbb{R}$ . A theorem of Hölder asserts that  $\tilde{G}_1$  is abelian. See [N] for a short proof, or [Th] for an even shorter proof. The canonical projection  $\pi : \tilde{G}_1 \rightarrow G_1$  is a group homomorphism, and  $G_1 = \pi(\tilde{G}_1)$  would be abelian, contrary to the assumption of the proposition.

Since  $G_1$  and  $G_2$  commutes, the fixed point set  $\text{Fix}(g)$  of any element  $g \in F_1$  is  $G_2$  invariant. Therefore we have

$$(1) \quad X_2 \subset \text{Fix}(g) \text{ for any } g \in F_1.$$

This shows that  $F_1$  is in fact a subgroup. By the very definition,  $F_1$  is normal. Since  $G_1$  is simple and  $F_1$  is nontrivial,  $F_1 = G_1$ . Finally again by (1),  $\text{Fix}(G_1) \neq \emptyset$ . Then the minimal set of  $G_1$  must be a singleton. Since  $G_2$  is conjugate to  $G_1$ , the minimal set  $X_2$  of  $G_2$  is also a singleton, contrary to the assumption.  $\square$

### 3. Fixed point set of $\Phi(G)$

Again consider  $\mathcal{G} = \text{Diff}_c^r(\mathbb{R}^n)_0$ , where  $n \geq 1$  and  $r \neq n + 1$ . We shall show Theorem 1.5 for the target group  $\text{Diff}^1(\mathbb{R})$  by a contradiction. So let us assume that  $\Phi : \mathcal{G} \rightarrow \text{Diff}^1(\mathbb{R})$  is a nontrivial homomorphism. By the simplicity of  $\mathcal{G}$ ,  $\Phi$  is injective and its image is contained in  $\text{Diff}_+^1(\mathbb{R})$ . For the purpose of showing Theorem 1.5, it is no loss of generality to assume the following.

**ASSUMPTION 3.1.** There is no global fixed point of  $\Phi(\mathcal{G})$ :  $\text{Fix}(\Phi(\mathcal{G})) = \emptyset$ .

In fact, we only have to pass from  $\mathbb{R}$  to a connected component of  $\mathbb{R} \setminus \text{Fix}(\Phi(\mathcal{G}))$ . This assumption will be made all the way until the end of the proof of Theorem 1.5.

We consider an embedding of Baumslag-Solitar group  $BS(1, 2)$  into the group  $\mathcal{G}(B)$ . See Section 2 for the definition of  $\mathcal{G}(B)$ . Recall that

$$BS(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle.$$

This group is a subgroup of  $GA$ , the group of the orientation preserving affine transformations of  $\mathbb{R}$ , where  $a$  corresponds to  $x \mapsto 2x$ , and  $b$  to  $x \mapsto x + 1$ . The group  $GA$  is a subgroup of  $PSL(2, \mathbb{R})$ . The group  $PSL(2, \mathbb{R})$  acts on the circle at infinity  $S_\infty^1$  of the Poincaré upper half plane, where  $GA$  is the isotropy subgroup of  $\infty \in S_\infty^1$ . Cutting  $S_\infty^1$  at  $\infty$ , we get a  $C^\infty$  action of  $BS(1, 2)$  on a compact interval, say  $[-1, 1]$ . This is called the *affine* action of  $BS(1, 2)$ .

T. Tsuboi [Ts] showed that there is a homeomorphism  $h$  of  $[-1, 1]$  which is a  $C^\infty$  diffeomorphism on  $(-1, 1)$  such that the conjugate by  $h$  of any element of  $BS(1, 2)$  is  $C^\infty$  tangent to the identity at the end points. Then the conjugated action extends to an  $C^\infty$  action on  $[-2, 2]$  in such a way that it is trivial on  $[-2, -1] \cup [1, 2]$ . Consider a subset  $S^{n-1} \times [-2, 2]$  embedded in  $B$ . The group  $GA$  acts on  $S^{n-1} \times [-2, 2]$ , trivially on the first factor. This way we obtain a subgroup of  $\mathcal{G}(B)$  isomorphic to  $BS(1, 2)$ , which we shall denote by  $G$ .

The key fact for the proof of Theorem 1.5 is the following result of [BMNR], which improves a semiconjugacy result in [GL].

**THEOREM 3.2.** (C. Bonatti, I. Monteverde, A. Navas and C. Rivas) *Assume  $BS(1, 2)$  acts faithfully on a compact interval by  $C^1$  diffeomorphisms in such a way that there is no interior global fixed point. Then the action is topologically conjugate to the affine action. In particular, all the interior orbits are dense.*

The compactness assumption on the interval is indispensable. In fact, there is a  $C^\infty$  exotic action of  $BS(1, 2)$  on  $\mathbb{R}$ . See [CC]. Thus in order to apply the above theorem, we need the following fixed point result in the first place.

**PROPOSITION 3.3.** *The fixed point set  $\text{Fix}(\Phi(G))$  is nonempty.*

**PROOF.** We assume for contradiction that  $\text{Fix}(\Phi(G)) = \emptyset$ . The proof follows the same line as Proposition 2.2. But since our target manifold is  $\mathbb{R}$  and is noncompact, extra care will be needed.

Since  $\text{Fix}(\Phi(G)) = \emptyset$ , any orbit of  $\Phi(G)$  is unbounded towards both directions. Since  $G$  is finitely generated,  $\Phi(G)$  has a compact cross section  $I$  in  $\mathbb{R}$ , that is, a compact interval  $I$  which intersects any  $\Phi(G)$  orbit. In fact, choose any point  $x_0 \in \mathbb{R}$  and let  $x_1$  be the supremum of  $g(x_0)$ , where  $g$  runs over a finite symmetric generating set. Then clearly any orbit intersects the interval  $I = [x_0, x_1]$ . Since  $G \subset \mathcal{G}(B)$ ,  $I$  is also a cross section for  $\Phi(\mathcal{G}(B))$ . That is, any  $\Phi(\mathcal{G}(B))$  orbit intersects the compact interval  $I$ .

Now we follow the proof of Proposition 6.1 in [DKNP], to show that there is a unique minimal set  $X$  for  $\Phi(\mathcal{G}(B))$ . In fact we shall show a bit more: there is a nonempty  $\Phi(\mathcal{G}(B))$  invariant closed subset  $X$  in  $\mathbb{R}$  which has the property that any nonempty  $\Phi(\mathcal{G}(B))$  invariant closed subset contains  $X$ .

The proof goes as follows. Let  $F$  be the family of nonempty  $\Phi(\mathcal{G}(B))$  invariant closed subsets of  $\mathbb{R}$ , and  $F_I$  the family of nonempty closed subsets  $Y$  in  $I$  such that  $\Phi(\mathcal{G}(B))(Y) \cap I = Y$ , where we denote

$$\Phi(\mathcal{G}(B))(Y) = \bigcup_{g \in \mathcal{G}(B)} \Phi(g)(Y).$$

Define a map  $\phi : F \rightarrow F_I$  by  $\phi(X) = X \cap I$ , and  $\psi : F_I \rightarrow F$  by  $\psi(Y) = \Phi(\mathcal{G}(B))(Y)$ . They satisfy  $\psi \circ \phi = \phi \circ \psi = \text{id}$ .

Let  $\{Y_\alpha\}$  be a totally ordered set in  $F_I$ . Then the intersection  $\cap_\alpha Y_\alpha$  is nonempty. Let us show that it belongs to  $F_I$ , namely,

$$(2) \quad \Phi(\mathcal{G}(B))(\cap_\alpha Y_\alpha) \cap I = \cap_\alpha Y_\alpha.$$

For the inclusion  $\subset$ , we have

$$\Phi(\mathcal{G}(B))(\cap_\alpha Y_\alpha) \cap I \subset (\cap_\alpha \Phi(\mathcal{G}(B))(Y_\alpha)) \cap I = \cap_\alpha (\Phi(\mathcal{G}(B))(Y_\alpha) \cap I) = \cap_\alpha Y_\alpha.$$

For the other inclusion, notice that

$$\cap_{\alpha} Y_{\alpha} \subset \Phi(\mathcal{G}(B))(\cap_{\alpha} Y_{\alpha}) \text{ and } \cap_{\alpha} Y_{\alpha} \subset I.$$

Therefore by Zorn's lemma, there is a minimal element  $Y$  in  $F_I$ . The set  $Y$  is not finite. In fact, if it is finite, the set  $X = \psi(Y)$  in  $F$  is discrete, and there would be a nontrivial homomorphism from  $\Phi(\mathcal{G}(B))$  to  $\mathbb{Z}$ , contrary to the fact that  $\mathcal{G}(B)$ , and hence  $\Phi(\mathcal{G}(B))$ , is simple.

Now the correspondence  $\phi$  and  $\psi$  preserve the inclusion. This shows that there is no nonempty  $\Phi(\mathcal{G}(B))$  invariant closed proper subset of  $X = \psi(Y)$ . In other words, any  $\Phi(\mathcal{G}(B))$  orbit contained in  $X$  is dense in  $X$ . Therefore  $X$  is either  $\mathbb{R}$  itself or a locally Cantor set. In the former case, any nonempty  $\Phi(\mathcal{G}(B))$  invariant closed subset must be  $\mathbb{R}$  itself.

Let us show that in the latter case,  $X$  satisfies the desired property:  $X$  is contained in any nonempty  $\Phi(\mathcal{G}(B))$  invariant closed subset. For this, we only need to show that the  $\Phi(\mathcal{G}(B))$  orbit of any point  $x$  in  $\mathbb{R} \setminus X$  accumulates to a point in  $X$ . In fact, if this is true, then any nonempty  $\Phi(\mathcal{G}(B))$  invariant closed subset must intersect  $X$ . But the intersection must be the whole  $X$  by the above remark.

Let  $(a, b)$  be the connected component of  $\mathbb{R} \setminus X$  that contains  $x$ . (If  $x \in X$ , there is nothing to prove.) There is a sequence  $g_k \in \mathcal{G}(B)$  ( $k \in \mathbb{N}$ ) such that  $\Phi(g_k)(a)$  accumulates to  $a$  and that  $\Phi(g_k)(a)$ 's are mutually distinct. Then the intervals  $\Phi(g_k)((a, b))$  are mutually disjoint, and consequently  $\Phi(g_k)(x)$  converges to  $a$ . This concludes the proof that  $X$  is contained in any nonempty  $\Phi(\mathcal{G}(B))$  invariant closed subset.

Choose  $B' \in \mathcal{B}$  such that  $B' \cap B = \emptyset$ . Any element of  $\mathcal{G}(B')$  commutes with any element of  $\mathcal{G}(B)$ . Define  $\mathcal{F}(B')$  to be the subset of the group  $\mathcal{G}(B')$  consisting of those elements  $g$  such that  $\text{Fix}(\Phi(g)) \neq \emptyset$ . By a theorem of Hölder, there is a nontrivial element in  $\mathcal{F}(B')$ . For any  $g \in \mathcal{F}(B')$ , the set  $\text{Fix}(\Phi(g))$  is closed, nonempty and invariant by  $\Phi(\mathcal{G}(B))$  by the commutativity. Therefore we have

$$(3) \quad X \subset \text{Fix}(\Phi(g)) \text{ for any } g \in \mathcal{F}(B').$$

This shows that  $\mathcal{F}(B')$  is a subgroup of  $\mathcal{G}(B')$ , normal and nontrivial. Since  $\mathcal{G}(B')$  is simple, we have  $\mathcal{F}(B') = \mathcal{G}(B')$ . Finally again by (3), we get  $\text{Fix}(\Phi(\mathcal{G}(B')))) \neq \emptyset$ . Since  $\mathcal{G}(B)$  is conjugate to  $\mathcal{G}(B')$  and  $G$  is a subgroup of  $\mathcal{G}(B)$ , we have  $\text{Fix}(\Phi(G)) \neq \emptyset$ , contrary to the assumption. The contradiction concludes the proof of Proposition 3.3.  $\square$

#### 4. Fixed point set of $\Phi(\mathcal{G}_B)$

For  $B \in \mathcal{B}$ , define a subgroup  $\mathcal{G}_B$  of  $\mathcal{G}$  by

$$\mathcal{G}_B = \{g \in \mathcal{G} \mid g = \text{id in a neighbourhood of } B\}.$$

Let  $\Phi : \mathcal{G} \rightarrow \text{Diff}_+^1(\mathbb{R})$  be a homomorphism satisfying Assumption 3.1. The purpose of this section is to show the following.

**PROPOSITION 4.1.** *For any  $B \in \mathcal{B}$ , the fixed point set  $\text{Fix}(\Phi(\mathcal{G}_B))$  is nonempty.*

**PROOF.** Any element of  $\Phi(\mathcal{G}(B))$  commutes with any element of  $\Phi(\mathcal{G}_B)$ . Let us denote  $F = \text{Fix}(\Phi(G))$ , which we have shown to be nonempty in Proposition 3.3. Clearly  $F$  is invariant by any element of  $\Phi(\mathcal{G}_B)$ . We shall show that there is a fixed point of  $\Phi(\mathcal{G}_B)$  in  $F$ . If  $F$  is bounded to the left or to the right, then the

extremal point will be a fixed point of  $\Phi(\mathcal{G}_B)$ . So we assume that  $F$  is unbounded towards both directions. That is, any connected component  $U$  of  $\mathbb{R} \setminus F$  is bounded.

Assume that there is  $g \in \mathcal{G}_B$  such that  $\Phi(g)(U) \cap U = \emptyset$ . (Otherwise  $\Phi(g)(U) = U$  for any  $g \in \mathcal{G}_B$ , and the proof will be complete.) There is a subgroup  $G'$  of  $\mathcal{G}_B$  conjugate to  $G$ . By some abuse, denote the generators of  $G'$  by  $a$  and  $b$ . They satisfy  $aba^{-1} = b^2$ . Notice that finite products of conjugates of  $b^{\pm 1}$  by elements of  $\mathcal{G}_B$  form a normal subgroup of  $\mathcal{G}_B$ . Since  $\mathcal{G}_B$  is simple, any element of  $\mathcal{G}_B$  can be written as such a product. Writing the above element  $g$  this way, one finds a conjugate of  $b$  whose  $\Phi$ -image displaces  $U$ . We may assume that  $\Phi(b)U \cap U = \emptyset$ , passing from  $G'$  to its conjugate by an element of  $\mathcal{G}_B$  if necessary. (The conjugate is still denoted by  $G'$ .)

Let  $V$  be the component of  $\mathbb{R} \setminus \text{Fix}(\Phi(G'))$  that contains  $U$ . Since  $G'$  is conjugate to  $G$ ,  $V$  is a bounded open interval and  $F \cap V$  is a closed nonempty proper subset of  $V$  invariant by  $\Phi(G')$ . It is easy to show that  $\Phi(b)|_V \neq \text{id}$  implies that the action  $\Phi(G')|_V$  is faithful. By Theorem 3.2, any  $\Phi(G')$  orbit in  $V$  must be dense in  $V$ . This contradicts the fact that  $F \cap V$  is invariant by  $\Phi(G')$ . The proof is now complete.  $\square$

## 5. Proof of Theorem 1.5

Again we assume that  $\Phi : \mathcal{G} \rightarrow \text{Diff}_+^1(S^1)$  is a homomorphism satisfying Assumption 3.1. Our purpose here is to get a contradiction. We follow an argument in [Mi].

LEMMA 5.1. *Assume  $B$  and  $B'$  are mutually disjoint balls of  $\mathcal{B}$ . Then any  $g \in \mathcal{G}$  can be written as  $g = g_1 \circ g_2 \circ g_3$ , where  $g_1$  and  $g_3$  belongs to  $\mathcal{G}_B$  and  $g_2$  to  $\mathcal{G}_{B'}$ .*

PROOF. Take any  $g \in \mathcal{G}$ . Then there is an element  $g_1 \in \mathcal{G}_B$  such that  $g_1^{-1} \circ g(B)$  is disjoint from  $B'$ . Next, there is an element  $g_2 \in \mathcal{G}_{B'}$  such that  $g_2^{-1} \circ g_1^{-1} \circ g$  is the identity in a neighbourhood of  $B$ . Thus  $g_3 = g_2^{-1} \circ g_1^{-1} \circ g$  belongs to  $\mathcal{G}_B$  and the proof is complete.  $\square$

LEMMA 5.2. *Assume  $B$  and  $B'$  are mutually disjoint elements of  $\mathcal{B}$ . If two points  $a$  and  $b$  ( $a < b$ ) belong to  $\text{Fix}(\Phi(\mathcal{G}_B))$ , then  $\text{Fix}(\Phi(\mathcal{G}_{B'})) \cap [a, b] = \emptyset$ .*

PROOF. Assume a point  $c$  in  $[a, b]$  belongs to  $\text{Fix}(\Phi(\mathcal{G}_{B'}))$ . Choose an arbitrary element  $g \in \mathcal{G}$ . There is a decomposition  $g = g_1 \circ g_2 \circ g_3$  as in Lemma 5.1. Now  $\Phi(g_3)(a) = a$ . Since  $\Phi(g_2)(c) = c$  and  $a \leq c$ , we have  $\Phi(g_2) \circ \Phi(g_3)(a) \leq c$ . Likewise  $\Phi(g)(a) = \Phi(g_1) \circ \Phi(g_2) \circ \Phi(g_3)(a) \leq b$ . Since  $g \in \mathcal{G}$  is arbitrary, the  $\Phi(\mathcal{G})$  orbit of  $a$  is bounded from the right. Then the supremum of the orbit must be a global fixed point, which is against Assumption 3.1:  $\Phi(\mathcal{G})$  has no global fixed point.  $\square$

For any point  $x \in \mathbb{R}^n$ , define a subgroup  $\mathcal{G}_x$  of  $\mathcal{G}$  by

$$\mathcal{G}_x = \{g \in \mathcal{G} \mid g \text{ is the identity in a neighbourhood of } x\}.$$

LEMMA 5.3. *For any  $x \in \mathbb{R}^n$ , the fixed point set  $\text{Fix}(\Phi(\mathcal{G}_x))$  is nonempty.*

PROOF. Notice that for any  $x \in \mathbb{R}^n$ , there is an decreasing sequence  $\{B_k\}$  ( $k \in \mathbb{N}$ ) in  $\mathcal{B}$  such that  $\{x\} = \bigcap_k B_k$ . Then  $\mathcal{G}_{B_k}$  is an increasing sequence of subgroups of  $\mathcal{G}$  such that  $\bigcup_k \mathcal{G}_{B_k} = \mathcal{G}_x$ . Therefore the closed subsets  $\text{Fix}(\Phi(\mathcal{G}_{B_k}))$

is decreasing and we have

$$\text{Fix}(\Phi(\mathcal{G}_x)) = \bigcap_k \text{Fix}(\Phi(\mathcal{G}_{B_k})).$$

Now it suffices to prove that  $\text{Fix}(\Phi(\mathcal{G}_B))$  is compact for  $B \in \mathcal{B}$ . Recall that we have already shown that  $\text{Fix}(\Phi(\mathcal{G}_B))$  is nonempty. Assume in way of contradiction that  $\sup \text{Fix}(\Phi(\mathcal{G}_B)) = \infty$ . (The other case can be dealt with similarly.) Choose  $B' \in \mathcal{B}$  such that  $B \cap B' = \emptyset$ . Notice that  $\Phi(\mathcal{G})$  consists of orientation preserving diffeomorphisms and  $\Phi(\mathcal{G}_{B'})$  is conjugate to  $\Phi(\mathcal{G}_B)$  by such a diffeomorphism. Therefore we also have that  $\sup \text{Fix}(\Phi(\mathcal{G}_{B'})) = \infty$ . Now one can find points  $a, b \in \text{Fix}(\Phi(\mathcal{G}_B))$  and a point  $c \in \text{Fix}(\Phi(\mathcal{G}_{B'}))$  such that  $a < c < b$ . This is contrary to Lemma 5.2.  $\square$

We use the assumption  $n \geq 2$  only in the sequel. Let  $D_0$  be the unit compact disc centered at 0 in  $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ . Define a family  $\mathcal{D}$  of closed subsets of  $\mathbb{R}^n$  by

$$\mathcal{D} = \{g(D_0) \mid g \in \mathcal{G}\}.$$

For any  $D \in \mathcal{D}$ , define a subgroup  $\mathcal{G}_D$  of  $\mathcal{G}$  by

$$\mathcal{G}_D = \{g \in \mathcal{G} \mid g \text{ is the identity in a neighbourhood of } D\}.$$

Lemma 5.3 implies that  $\text{Fix}(\Phi(D)) \neq \emptyset$  for any  $D \in \mathcal{D}$ .

LEMMA 5.4. *For any  $D \in \mathcal{D}$ , the set  $\text{Fix}(\Phi(D))$  is a singleton.*

PROOF. First of all notice that for any  $D, D' \in \mathcal{D}$  such that  $D \cap D' = \emptyset$ , we have  $\text{Fix}(\Phi(\mathcal{G}_D)) \cap \text{Fix}(\Phi(\mathcal{G}_{D'})) = \emptyset$ . In fact, as is easily shown,  $\mathcal{G}_D$  and  $\mathcal{G}_{D'}$  generate  $\mathcal{G}$ . Thus the point of the above intersection would be a global fixed point of  $\mathcal{G}$ , against Assumption 3.1. This shows that the interior  $\text{Int}(\text{Fix}(\Phi(\mathcal{G}_D)))$  is empty. In fact, there are uncountably many mutually disjoint elements of  $\mathcal{D}$ , while mutually disjoint open subsets of  $\mathbb{R}$  are at most countable.

Assume that  $\text{Fix}(\Phi(\mathcal{G}_D))$  contains more than one points. Since  $\text{Int}(\text{Fix}(\Phi(\mathcal{G}_D)))$  is empty,  $\text{Fix}(\Phi(\mathcal{G}_D))$  is not connected. To any  $D \in \mathcal{D}$ , assign a bounded component  $I_D$  of  $\mathbb{R} \setminus \text{Fix}(\Phi(\mathcal{G}_D))$  in an arbitrary way. This is possible by the axiom of choice. Notice that Lemmata 5.1 and 5.2 for the family  $\mathcal{B}$  are valid for  $\mathcal{D}$  as well. (No changes of the proofs are needed.) Consequently  $I_D \cap I_{D'} = \emptyset$  if  $D \cap D' = \emptyset$ . Again this is contrary to the fact that there are uncountably many mutually disjoint elements of  $\mathcal{D}$ .  $\square$

Finally let us prove Theorem 1.5. Choose any element  $D \in \mathcal{D}$  and distinct two points  $x_1, x_2 \in D$  that are contained in  $D$ . Then since  $\text{Fix}(\Phi(\mathcal{G}_D))$  is a singleton and  $\text{Fix}(\Phi(\mathcal{G}_{x_i}))$  is nonempty, we have  $\text{Fix}(\Phi(\mathcal{G}_{x_1})) = \text{Fix}(\Phi(\mathcal{G}_{x_2}))$ . But  $\mathcal{G}_{x_1}$  and  $\mathcal{G}_{x_2}$  generate  $\mathcal{G}$ , and there would be a global fixed point of  $\Phi(\mathcal{G})$ , against Assumption 3.1. The contradiction shows that the homomorphism  $\Phi$  must be trivial.

## 6. Sporadic results for $\text{Diff}^0(S^1)$ target

Let  $M = L \times S^m$  be a closed  $n$ -dimensional manifold such that  $1 \leq m \leq n$ . Then we have the following result.

THEOREM 6.1. *If  $n \geq 2$  and  $r \neq n + 1$ , there is no nontrivial homomorphism from  $\text{Diff}_c^r(M)_0$  to  $\text{Diff}^0(S^1)$ .*

PROOF. Assume that  $\Phi : \text{Diff}_c^r(M)_0 \rightarrow \text{Diff}^0(S^1)$  is a nontrivial homomorphism. The Lie group  $PSL(2, \mathbb{R})$  acts on  $S^m$  as Moebius transformations. So it acts on  $M = L \times S^m$ , trivially on  $L$ -coordinates. Denote the inclusion by  $\iota : PSL(2, \mathbb{R}) \rightarrow \text{Diff}_c^r(M)_0$ . The simplicity of the group  $\text{Diff}_c^r(M)_0$  shows that the homomorphism

$$\Phi \circ \iota : PSL(2, \mathbb{R}) \rightarrow \text{Diff}^0(S^1)$$

is nontrivial.

Now Theorem 5.2 in [Ma2] asserts that the homomorphism  $\Phi \circ \iota$  is the conjugation of the standard embedding  $\iota_0 : PSL(2, \mathbb{R}) \rightarrow \text{Diff}^0(S^1)$  by a homeomorphism of  $S^1$ . It is no loss of generality to assume that  $\Phi \circ \iota = \iota_0$ , by changing  $\Phi$  if necessary. If the dimension of  $L$  is positive, then  $\text{Diff}_c^r(L)_0$  also acts on  $M$ , trivially on  $S^m$ -coordinates. Any element of the group  $\Phi(\text{Diff}_c^r(L))$  must commute with any element of  $PSL(2, \mathbb{R})$ . But there is no nontrivial element in  $\text{Diff}_+^0(S^1)$  which commutes with all the element of  $PSL(2, \mathbb{R})$ . A contradiction.

Let us consider the case where  $L$  is a singleton. Then there is an element  $g$  in  $\text{Diff}^r(S^n)$ ,  $n \geq 2$ , which commutes with all the elements of  $\iota(SO(2))$  and is not contained in  $\iota(SO(2))$ . But  $\Phi \circ \iota(SO(2)) = SO(2)$ , and any element in  $\text{Diff}^0(S^1)$  which commutes with all the elements of  $SO(2)$  must be an element of  $SO(2)$ , contradicting the injectivity of  $\Phi$ .

REMARK 6.2. There are a wider class of manifolds for which the above argument holds. For example, if  $M$  is the unit tangent bundle of a closed hyperbolic surface and if  $r \neq 4$ , then any homomorphism from  $\text{Diff}_c^r(M)_0$  to  $\text{Diff}^0(S^1)$  is trivial.

## References

- [BMNR] C. Bonatti, I. Monteverde, A. Navas and C. Rivas, *Rigidity for  $C^1$  actions on the interval arising from hyperbolicity I: solvable groups*, arXiv:1309.5277.
- [CC] J. Cantwell and L. Conlon, *An interesting class of  $C^1$  foliations*, Topology and its applications **126**(2002), 281-197.
- [DKNP] B. Deroin, V. Kleptsyn, A. Navas and K. Parwani, *Symmetric random walks on  $\text{Homeo}^+(\mathbb{R})$* , Ann. Probability **41**(2013), 2066-2089.
- [G] É. Ghys, *Prolongements des difféomorphismes de la sphère*, L'Enseign. Math. **37**(1991), 45-59.
- [GL] N. Guelman and I. Lioussé,  *$C^1$ -actions of Baumslag-Solitar groups on  $S^1$* , Algebraic & Geometric Topology **11**(2011), 1701-1707.
- [M] K. Mann, *Homomorphisms between diffeomorphism groups*, To appear in Ergod. Th. Dyn. Sys.
- [Ma1] S. Matsumoto, *Numerical invariants for semiconjugacy of homeomorphisms of the circle*, Proc. A. M. S. **98**(1986), 163-168.
- [Ma2] S. Matsumoto, *New proofs of theorems of Kathryn Mann*, to appear in Kodai Math. J.
- [Mi] E. Milton, *Actions of groups of homeomorphisms on one-manifold*, Arxiv: 1302.3737.
- [N] A. Navas, *On the dynamics of (left) orderable groups*, Ann. l'inst. Fourier **60**(2010), 1685-1740.
- [Th] W. P. Thurston, *Three-manifolds, foliations and circles, I*, Arxiv:9712268.
- [Ts] T. Tsuboi,  *$\Gamma_1$ -structures avec une seule feuille*, Astérisque **116**(1984), 222-234.

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